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LETTER TO THE EDITOR

Lie symmetries for the reduced three-wave interaction problem

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Abstract. We show how the method of Lie symmetries can be used to obtain first integrals and for the identification of completely or partially integrable cases of the reduced three-wave interaction problem.

The analysis of regular or chaotic behaviour of general nonlinear systems is an important problem in applied mathematics. If the system is an integrable one the solutions will be well behaved and we can get global information on the long-term behaviour of the system. The notion of integrability is related to the existence of first integrals. Therefore, an important question to ask is: given a system of ordinary differential equations, depending on parameters, how can we identify the values of the parameters for which the equations have first integrals? Several methods have been employed for studying the existence of first integrals and the integrability of dynamical systems. Some of them have been devised for Hamiltonian systems, such as the Ziglin–Yoshida analysis [1] or the method of Noether symmetries [2]. Other methods can be applied also for non-Hamiltonian systems: the direct method [3], the singularity analysis [4], the linear compatibility analysis method [5], the use of Lax pairs [6], the method of Lie symmetries [7, 8], the quasimonomial formalism [9], the Carlemann embedding procedure [10], etc. Usually the systems considered are three-dimensional systems of first-order differential equations or two-dimensional Hamiltonian systems, with a four-dimensional phase space.

We use here the method of Lie symmetries for studying a specific three-dimensional model: the reduced three-wave interaction problem [11]. The Painlevé analysis of this system was made by Bountis *et al* [12]; they found first integrals for several values of the parameters and identified integrable and partially integrable cases. The symmetry method, introduced by Lie [13], consists of a systematic procedure for the determination of the symmetry transformations of the equations. By using these symmetries we can find first integrals in a straightforward fashion and identify integrable cases. This method was applied by Sen and Tabor [14] for an extensive analysis of the Lorenz model. Recently Giacomini *et al* [15] discussed the identification of first integrals for the three-wave interaction problem by using the direct method, with an ansatz of the form $I = e^{at}F(x, y, z)$, up to second order in z. The Lie symmetry method considered here permits also the obtention of all first integrals found by the direct method.

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We summarize here, without proofs, some results about the relationships between Lie symmetry vector fields and first integrals, for a system of first-order differential equations. They can be easily extended for nth-order differential equations.

The symmetry vector fields can be obtained from the invariance of the system of differential equations

$$\Delta_i(x_j, x_j') = 0 \tag{1}$$

under infinitesimal transformations with the form

$$\begin{aligned} x_i &\Rightarrow x_i + \varepsilon\eta_i(t, x_i) \\ t &\Rightarrow t + \varepsilon\xi(t, x_i). \end{aligned} \tag{2}$$

We will take $\xi = 0$. The symmetry evolutionary vector field has the form

$$U = \eta_i \partial_{x_i}. \tag{3}$$

The Lie conditions for the invariance of the system (1) are

$$pr^{(1)}U(\Delta_i)|_{\Delta_i=0} = [\eta_j\partial_{x_j} + (D_t\eta_j)\partial_{x_j'}](\Delta_i)|_{\Delta_i=0} = 0. \tag{4}$$

If I is a time-dependent first integral for the system (1) then

$$D_t I = \partial_t I + X(I) = 0 \tag{5}$$

where X is the dynamical vector field.

Theorem A. Given a set of first-order differential equations, if U_i is a symmetry vector field of Δ_i then $U_2 = IU_1$ is also a symmetry vector field of Δ_i, if and only if I is a first integral of the system Δ_i.

In particular, if a symmetry vector field U_1 is not functionally independent of the dynamical vector field X,

$$U_1 = F(t, x_i)X \tag{6}$$

then F is a first integral of the system.

We will look for symmetries of the following three-dimensional model:

$$\begin{aligned} \Delta\colon x' &= ax + by + z - 2y^2 \\ y' &= ay - bx + 2xy \\ z' &= -2z - 2zx. \end{aligned} \tag{7}$$

This system of differential equations corresponds to the interaction of three quasi-synchronous waves in a plasma with quadratic nonlinearities [11]. Making the change of variables

$$y \Rightarrow y - b/2 \tag{8}$$

the equations (7) are rewritten in the form

$$\begin{aligned} x' &= ax - by + z - 2y^2 \\ y' &= ay + ab/2 + 2xy \\ z' &= -2z - 2xz. \end{aligned} \tag{9}$$

By using the Lie method we analysed the invariance of the equations (9) under the infinitesimal transformations (2). We supposed a polynomial dependence of the functions η_i up to the second order in z. The calculations of the symmetry vector fields were made by algebraic computation. We give here the final results for the parameter values at which non-trivial symmetries exist.

(A) $a=0$, b arbitrary. From the equations (9) and (4), the following symmetry vector fields are found:

$$\begin{aligned} U_1 &= (-2y^2+z-by)\partial_x + 2xy\partial_y - (2z+2zx)\partial_z = X \\ U_2 &= zy\, e^{2t} X. \end{aligned} \tag{10}$$

The integral of motion

$$I = zy\, e^{2t} \tag{11}$$

is easily found, by using theorem A. The equations (9), in this case, possess the Painlevé property [12] and can be reduced to the equation

$$y'' = y'^2/y - 4y^3 + 2\, e^{-2t} - 2by^2. \tag{12}$$

The point transformation $Y = e^t y$, $T = e^{-t}$ reduces (12) to the third Painlevé transcendent.

(B) $b=0$, a arbitrary. Solving (4) we get the symmetry vector fields

$$\begin{aligned} U_1 &= (z+ax-2y^2)\partial_x + (2xy+ay)\partial_y - (2zx+2z)\partial_z = X \\ U_2 &= zy\, e^{(2-a)t} X \end{aligned} \tag{13}$$

and

$$I = zy\, e^{(2-a)t} \tag{14}$$

is the first integral of the system. It was identified by Kruskal *et al* [16]. The Painlevé conditions are partially fulfilled in this case.

(C) $a=-1$, b arbitrary. The symmetry vector fields, in this case, are

$$\begin{aligned} U_1 &= (z-x-2y^2-by)\partial_x + (2xy-y-b/2)\partial_y - (2zx+2z)\partial_z = X \\ U_2 &= (x^2+y^2+z+by+b^2/4) X. \end{aligned} \tag{15}$$

The first integral is

$$I = e^{2t}(x^2+y^2+z+by+b^2/4). \tag{16}$$

There are logarithmic terms in the singular expansion of the variables, therefore the equations (9), for these values of the parameters, do not have the Painlevé property. This system is an example of a partially integrable system [12].

(D) $a=-1$, $b=0$. The symmetry vector fields are

$$\begin{aligned} U_1 &= (z-x-2y^2)\partial_x + (2xy-y)\partial_y - (2zx+2z)\partial_z = X \\ U_2 &= e^{2t}(z+x^2+y^2) X \\ U_3 &= e^t[(z-2y^2)\partial_x + 2xy\partial_y - 2xz\partial_z] \\ U_4 &= zy\, e^{3t} U_3 \\ U_5 &= e^{2t}(z+x^2+y^2) U_3 \\ U_6 &= zy\, e^{3t} X. \end{aligned} \tag{17}$$

Theorem A leads to the following first integrals:

$$I_1 = zy\, e^{3t} \qquad I_2 = (z+x^2+y^2)\, e^{2t}. \tag{18}$$

This case is completely integrable by the use of these first integrals.

(E) $a=-2$, b arbitrary. We get, from (4) and (9) the following symmetry vector fields:

$$\begin{aligned} \boldsymbol{U}_1 &= (z-2x-2y^2-by)\partial_x + (2xy-2y-b)\partial_y - (2zx+2z)\partial_z = \boldsymbol{X} \\ \boldsymbol{U}_2 &= \mathrm{e}^{4t}[2zy+b(z+x^2+y^2)+b^2(y+b/4)]\boldsymbol{X} \end{aligned} \tag{19}$$

and the first integral is

$$I = \mathrm{e}^{4t}[2zy+b(z+x^2+y^2)+b^2(y+b/4)]. \tag{20}$$

The cases considered above exhaust the situations where there is a non-trivial Lie symmetry up to the second order in z. It appears unlikely that there are more cases for further values of the parameters. We note that in [15] there is a mistake when the authors make a remark about misprints in the formulae (3.7) and (3.11) of [12]. There are no such misprints. They overlook the fact that Bountis *et al* used, as we did in this letter, the transformed equations (9) and not the original system (7).

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